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Article

Application of functional analysis in signal processing

Shuo Wang*

College of Information Science and Engineering, Zaozhuang University, Zaozhuang 277160, China
* Correspondence: wswx828211@163.com

Abstract: This paper presents the application of linear generalization in the field of digital signal processing. Digital signal processing, as a common signal processing method, requires the digital conversion of analog signals. In order to better process digital signals, mathematical tools such as matrix operations must be used. This is where linear generalization becomes particularly important. A linear general function is a linear mapping from a vector space to a corresponding pure volume domain and can be expressed as the action of a column vector on a vector. The processing of linear general functions enables the finer handling of digital signals, facilitating the filtering out of noise and interference, and thus the generation of higher-quality signals. Furthermore, the linear general function is employed in a variety of signal recognition, feature extraction, peak detection and other applications, providing a robust theoretical foundation for digital signal processing.

Keywords: Digital signal processing; Signal space; Extremum; Linear functional

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1. Introduction

The general analytical approach in digital signal processing is grounded in functional theory, integrating mathematical tools such as linear algebra, differential equations, and integral transforms to explore the intrinsic properties of signals. While this methodology is widely regarded for its rigorous mathematical logic and clear physical concepts, and is recognized as an excellent analytical framework [1-2], it is often limited by its inability to offer a comprehensive generalization. Various transformations remain disconnected, highlighting the approach's inherent constraints. Consequently, alternative methods are necessary to facilitate a more in-depth investigation of signal behavior.

Functional analysis, as a cornerstone of modern mathematical analysis, not only enhances our understanding of the intrinsic structure of functions and function spaces but also bridges the gap between abstract mathematical theory and practical scientific applications. By extending the concept of vector spaces to infinite dimensions—namely, function spaces—functional analysis investigates the interactions of functions as elements within these spaces and their behavior under specific operations. This encompasses profound concepts such as the rigorous metrics of metric spaces, the distance and completeness in normed spaces, the symmetry and positive definiteness in inner product spaces, and the completeness and orthogonality in Hilbert spaces. Although the highly abstract nature of functional analysis may appear distant from real-world problems, it plays a pivotal role in various fields, including calculus solving, quantum mechanics analysis, statistical inference, and signal processing.

In the field of signal processing, traditional signals are no longer mere time sequences or numerical arrays, but are redefined as deeper mathematical entities—vectors in infinite-dimensional function spaces. This shift in perspective has not only simplified the description and analysis of signal characteristics but has also reduced the complexity of the problem. Signal processing systems, in turn, are no longer confined to a set of algorithms or hardware implementations. Instead, they are abstracted as linear operators or, more broadly, transformations acting on the signal space, capable of capturing and transforming the essential features of signals, whether for tasks such as filtering, compression, or pattern recognition. The integration of theory and practice goes beyond theoretical derivation and demonstrates significant potential and value in real-world applications of digital signal processing. For instance, in communication system design, functional analysis methods enable more precise filter design to eliminate noise while preserving signal integrity; in image processing, it facilitates the development of efficient image compression algorithms that reduce data storage and transmission burdens without loss of quality; and in speech recognition and natural language processing, the principles of functional analysis underpin the extraction of key speech signal features, enhancing intelligent interaction experiences. This paper aims to elucidate the applications of functional analysis in the field of digital signal processing by integrating its fundamental theory with relevant knowledge in the domain.

2. Fundamental Theory

2.1. Functional Sets and Signal Spaces

In functional analysis, the definition of a set refers to the collection of entities that possess specific properties or satisfy certain conditions, with each entity being termed an element of the set [3-5]. In signal processing, each group of discrete signals can be represented by a set, which we refer to as the signal space. For example, the set of periodic sine signals can be expressed as:

$$S = \{X; X(T) = \psi_m[Ae^{j(\omega t+\theta)}];\ \ A, \omega, \theta \in R\} \tag{1}$$

The R^n space, or n-dimensional real space, is formed by discrete-time sequences consisting of n sample points. The $C(T)$ space, or continuous-time space, is composed of continuous-time signals. The $L^2(T)$ space, or square-integrable space, consists of signals with finite energy.

The n-dimensional real space is constructed from discrete-time sequences, while the continuous-time space or square-integrable space can be formed from continuous-time signals.

2.2. Extremum Problems in Signal Processing

Extremum problems are frequently encountered in signal processing. In signals, local maxima and minima often represent critical information. For instance, in speech recognition, a local maximum can correspond to the stress or emphasis in speech, while a local minimum may indicate the boundaries of the speech. In image processing, extrema highlight essential features such as object contours and edges. Therefore, accurately identifying the extrema within a signal becomes a fundamental task. The common approach to solving this problem involves differentiation to locate the points where the derivative of the function is either zero or undefined. These points are then analyzed using the second derivative test to classify them as maxima or minima.

Extremum problems pertain to the optimization of signals and systems, aiming to either maximize or minimize a specific functional or quadratic functional of the signal or system.

In a normed linear space, the directional derivative of functional $f(\bar{x})$ refers to the change in x_0, the directional derivative of $f(\bar{x})$ exists and equals zero in every direction, i.e., $D_u f(\bar{x}) = \bar{0}$, then this point is called a fixed point or an extremum point. This indicates that the value of the functional remains nearly invariant in the neighborhood of this point and is not affected by small perturbations. Such points hold significant importance in functional analysis and often serve as key points for optimization problems and stability analyses.

It can be demonstrated that the inner product can represent the directional derivatives of both linear functionals and quadratic functionals. That is,

$$D_u f(\bar{x}) = (\bar{u}, \bar{\varphi}) \tag{2}$$

$$D_u q(\bar{x}) = (A^* \bar{u}, \bar{u}) + (A\bar{x}, \bar{u}) \tag{3}$$

where A^* denotes the adjoint operator of A.

It is also proven that, in a real space, the inner product of the gradients (denoted as ∇f and ∇q of a linear functional and a quadratic functional can represent the directional derivatives of the two functionals, i.e.,

$$D_u f(\bar{x}) = (\nabla f, \bar{u}) \text{ and } D_u q(\bar{x}) = (\nabla q, \bar{u}) \tag{4}$$

From the above equation, we deduce that:

$$\nabla f = \bar{\varphi} \quad \text{and} \quad \nabla q = (A^* + A)\bar{x} \tag{5}$$

Therefore, the existence of a fixed point for the linear or quadratic functional of a signal constitutes a necessary condition for the signal to have an extremum, i.e., $\nabla f = \bar{0}$ or $\nabla q = \bar{0}$. In practical applications, the extremum problem of a signal may also be subject to an additional constraint, such as $f_0(\bar{x})$ =constant}. This is equivalent to seeking a fixed point within the subset satisfying this constraint, that is, finding the fixed point of $f(\bar{x}) + \lambda f_0(\bar{x})$. Thus, we have

$$D_u f(\bar{x}_0) + \lambda D_u f_0(\bar{x}_0) = \bar{0} \tag{6}$$

where λ is a constant, and the above equation can be represented in gradient form as:

$$\nabla f + \lambda \nabla f_0 = \bar{0} \tag{7}$$

solving the above equation yields the desired signal.

3. Application Examples

3.1 Fourier Transform

The Fourier transform is a linear integral transform that converts complex time-domain signals into frequency-domain representations, known as the signal's spectrum [6]. By processing the signal in the frequency domain, its features and structure can be analyzed more efficiently. The inverse Fourier transform, on the other hand, allows the processed frequency-domain signal to be converted back into a time-domain signal for further analysis or application. After processing in the frequency domain, the Fourier inverse transform can be used to revert these frequency-domain signals to their original time-domain form. The widespread application of Fourier transforms lies in the analysis and processing of various types of signals, such as audio and images. Frequency-domain analysis, by providing information on the signal's frequency components, helps to understand the signal's periodicity, frequency characteristics, and filtering operations. Additionally, by using the inverse Fourier transform, signals processed in the frequency domain can be reverted to their original time-domain form for further processing and application.

$$F(j\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \tag{8}$$

$$f(t) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{j\omega t} d\omega \tag{9}$$

A fundamental prerequisite for the existence of the Fourier transform of a given real-valued function f(t) is that the function must be integrable over the entire time axis.

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty \tag{10}$$

The Discrete Fourier Transform (DFT), as a commonly used foundational tool in signal analysis and processing, experiences a quadratic increase in computational complexity with the length of the sequence. This presents a significant computational barrier in large-scale data processing scenarios. To address this, the concept of the Fast Fourier Transform

(FFT) was introduced, aiming to dramatically reduce the computational burden of the DFT and enable high-speed processing of signal spectrum analysis. The core innovation of the FFT algorithm lies in its clever use of the fact that a signal sequence of length N can be recursively divided into several shorter subsequences, each of which undergoes DFT computation independently. The essence of this process is rooted in exploiting the periodic repetition and symmetry properties of the complex exponential terms in the DFT formula. Specifically, the FFT divides the original sequence into two halves, recursively converting the large problem into smaller ones until it reaches the basic units. Then, by combining and reassembling the DFT results of these basic units in a specific manner, the entire sequence's DFT is efficiently restored. This approach not only reduces the number of multiplication operations required but also transforms many computations into simple addition and subtraction, greatly enhancing the computational efficiency and practicality of the algorithm.

Let the signal model be:

$$x = 5 + 7\cos(2\pi\times 15t - 30\pi/180) + 3\cos(2\pi\times 40t - 90\pi/180) \tag{11}$$

From $Fn = (n-1) * Fs/N$, it is evident that the spacing between any two points is 0.5 Hz. This simulation is divided into three signal frequency bands: 0 Hz, 15 Hz, and 40 Hz. Figure 1 shows the Matlab simulation results, where (a) represents the original signal waveform, and (b) displays the amplitude spectrum and phase spectrum after the Fourier transform.

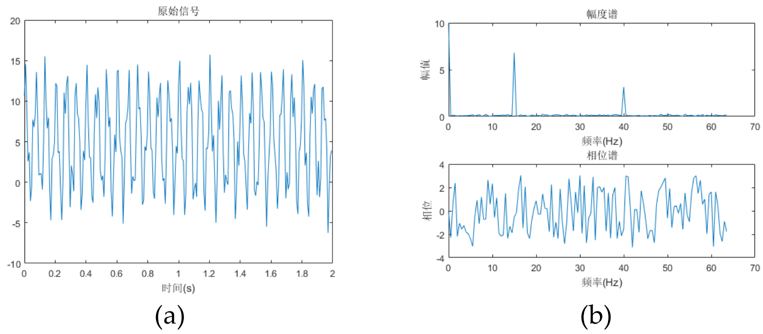

Figure 1. Fourier Transform Simulation Results

3.1 Fourier Transform

The Hilbert transform [7] is defined as follows:

$$\hat{s}(t) = H\{s\} = h(t) * s(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s(\tau)}{t-\tau}d\tau \tag{12}$$

Given a continuous-time signal f(t) , its analytic signal is defined as:

$$z(t) = f(t) + jf(t) \tag{13}$$

The Fourier transform of the analytic signal is:

$$Z(j\omega) = \begin{cases} 2F(j\omega) & \omega > 0 \\ 0 & \omega < 0 \end{cases} \tag{14}$$

From the Fourier transform, we observe that an even-symmetric real signal exhibits the phenomenon of frequency conjugation. That is, in the frequency spectrum, we observe components of a two-sided spectrum, where the positive frequency part has physical significance, while the negative frequency part does not. Therefore, in signal analysis, we discard the negative frequency component. However, negative frequencies do carry energy, so we need to transfer this energy to the positive frequency part. The Hilbert transform achieves this by converting the signal into an analytic signal, which is a complex signal, and then applying the Fourier transform to obtain the one-sided frequency spectrum.

Figure 2 presents the simulation results of the Hilbert transform. In Figure 2(a), the blue waveform represents the original signal, the orange waveform denotes the real part of the signal after Hilbert transform, and the yellow waveform stands for the imaginary part of the Hilbert-transformed signal. Figure 2(b) shows a locally enlarged comparison, from which it can be observed that the real part of the signal after Hilbert transform is exactly the original signal, while the imaginary part corresponds to the analytic signal—this imaginary part exhibits a phase shift relative to the real part. As illustrated in Figure 2(c), the signal after Hilbert transform exhibits a 90° phase shift effect. Figure 2(d) is the single-sided spectrum generated by the Hilbert transform: the orange waveform represents the Fourier transform, and the blue waveform represents the Hilbert transform. It can be seen that the Hilbert transform supplements the negative frequency components in the Fourier transform to the positive frequency range without causing energy loss.

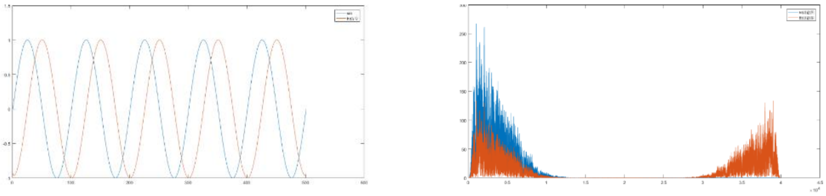

Figure 2. Hilbert Transform Simulation Results

3.3 LMS Adaptive Filtering

The LMS (Least Mean Squares) algorithm is an adaptive filtering technique based on the Wiener filtering principle, optimized through the method of steepest descent to minimize the mean square error between the filter output and the desired value. Especially in the absence of prior statistical knowledge of the input process, the LMS algorithm relies on observed data to continuously adjust the filter's parameters, learning during the adjustment process to gradually achieve the optimal filtering result. Therefore, the LMS algorithm is well-suited for processing non-stationary random signals.

Specifically, the LMS algorithm consists of three steps: First, the step size factor μ and the number of filter taps M are determined, and the parameters are initialized. Next, the LMS filter output is computed using the steepest descent method:

$$y(n) = \sum_{i=0}^{M-1} \omega_i x(n-i) \tag{15}$$

where $y(n)$ represents the LMS filtered signal, and $x(n)$ represents the input signal at time n. The mean square error $e(n)$ is then obtained based on the difference between the observed data and the desired value $d(n)$:

$$e(n) = d(n) - y(n) \tag{16}$$

The LMS algorithm updates the weight coefficients iteratively using the recursive formula:

$$\omega_k(n) = \omega_k(n-1) + \mu e(n) x(n) \tag{17}$$

where μ represents the step size factor of the LMS algorithm, which determines the stability and convergence rate of the system. Finally, the weight coefficients are updated iteratively using the LMS recursive formula to gradually stabilize the system and achieve the optimal filtering result. Figure 3 shows the simulation results of the LMS algorithm.

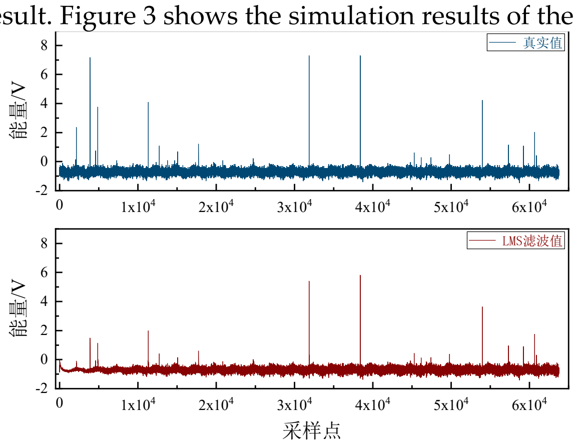

Figure 3. The simulation results of the LMS algorithm

4. Conclusions

The combination of functional analysis and digital signal processing has resulted in a more refined signal analysis method, making the signal analysis process simpler and significantly optimizing the algorithmic analysis process for digital signal processing. This method has broad applications in future signal analysis.

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